USE OF LOWER PARTIAL MOMENTS IN THE ASSET ALLOCATION PROCESS

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Abstract

Over the past years before the world financial and economic turbulences, the Baltic States have been the fastest developing economies in Europe. The Baltic insurance industry (and the Latvian one as well) was a direct beneficiary of this economic miracle. In 2002 – 2007, the local insurance market in three Baltic States doubled in volume. After the booming years insurance business suffered from the economic downturn as the income from main business operations did not show sustainable growth and companies should gain extra income from investing activities in order to stay on the market, but due to the vulnerable financial markets the return on investment decreased. So the relevance of asset allocation problem gained extra attention in the particular industry. The main purpose of the current paper is providing the foundation for the development of the "new" portfolio model. The reader is going to be instructed on the essential aspects of the (μ,LPM)- portfolio model which, on the one side, enables its critical review, and on the other side, provides a platform for its later application in the practice of portfolio management. The paper is covering only the theoretical aspect of the topic. The research is concerned with the portfolio selection based on the downside risk and mean, which utilises risk measure corresponding with the risk understanding of the prevailing number of investors. As a consequence, by the portfolio optimisation based on the downside risk the chance to over-perform the reference point is not minimised as by the portfolio optimisation based on the variance.

Keywords: asset allocation, portfolio management, lower partial moments.

Introduction

Information provided by Latvian Insurance Association (2011) allows to come to the conclusion that Latvian insurance companies who have been affected by challenging market conditions in 2008 – 2009 are de-risking portfolios and shedding questionable lines of business. As insurers seek to identify new sources of capital, as well as they need to allocate capital effectively among product lines and business units. Analysing the asset allocation question (in the framework of portfolio management), the considerations lead to the portfolio theory whereby the appropriate question that is asked is how it could be substantially improved nowadays in order to develop a "better" portfolio model as the portfolio model of Markowitz, which laid the basic ideas of the modern theory, is being consequently criticised due to the subjectivity of preferences.

The proposal of downside risk measure`s use aroused in academic research at the same time as the Markowitz's portfolio model discussed in the previous chapter. Different to Markowitz, who based the theory on the maximisation of the investor`s expected utility, Roy (1952:433) declared, that a man who seeks advice about his action will not be grateful for the suggestion that he has to maximise expected utility, and instead of the expected utility maximisation, he proposed the concept of safety of principal, while a minimal acceptable return has to be set. Consequently, the investor prefers the portfolio with the smallest probability of falling below this disaster level.

The probability of disaster introduced by Roy plays also important role in the Kataoka's criterion (1963:181-196), who claimed that the portfolio, which achieves from available efficient portfolios on the (μ, σ)-efficient frontier the highest target return for a predetermined probability of disaster, has to be chosen. Also Telser (1955-1956:1-16) applied the probability of disaster, but the selected portfolio is expected to achieve the highest return for a given probability of failing to achieve a predetermined level.

Later on the ideas expressed by Roy, Kataoka and Telser was called as "safety first" principle. The approach was developed later by Leibowitz and Henriksson (1989:34-41) and named "shortfall risk", while the authors paid special attention to the asset-liability management, where not asset return but surplus is relevant, while the benchmark is defined as the liabilities and is assumed to randomly fluctuate. Follow also papers by Leibowitz, Kogelman and Bader (1992:28-37); Jaeger and Zimmermann (1996:64-74).

General overview of the literature on safety first is given in Albrecht (2004:1-16). Markowitz accepted the idea of downside risk, and suggested two measures: below-mean semivariance and below-target semivariance, both capturing squared return deviations below mean or target return. Nonetheless, the most important restriction of the portfolio semivariance proposed by Markowitz is that it depends on asset weights. In the later proposal, the co-movements between individual asset returns falling below the target are not quantified, and therefore risk diversification is not reflected in the portfolio optimisation (Markowitz et. al., 1993:307-317).

The new period of downside risk research began with the generalised concept of downside risk defined by the Lower Partial Moment1 (LPM) developed by Bawa (1975:95-121) and Fishburn (1977:116-126).

Bawa and Lindberg (1977:189-200) examined the downside risk diversification and proposed the measure of return co-movements below the target return. Bookstaber and Clarke's (1981:63-70) worked on optioned portfolios and discovered the necessity of the consideration of additional moments of return distribution. Asset pricing model in the generalised LPM-framework was developed by Harlow and Rao (1989:285-311). The relationship of the (μ, LPM) - portfolio model to the capital market theory was developed by Nawrocki (1996:1-11), who declared that portfolio management strategies should derive from the segmented market theory. Segmented markets generate non-normal return distributions and require the use of utility theory, thus, the (μ, LPM) - model is the decision model, because it does not assume normal distributions and allows different utility goals expressed by. The characteristics of the downside risk-optimised portfolios were most extensively tested by Nawrocki, whereat the most important result was that portfolio skewness can be managed through the LPM measure, since with the increasing degree of LPM the portfolio skewness increases; the size and composition of portfolios selected by the (μ, LPM) - optimal algorithm in comparison with the (μ, σ) -efficient portfolios, and the effect of different degrees of risk aversion on the expected performance of derived portfolios were tested.

Since the nineties the downside risk measures have been increasingly attracting practitioners, who have initiated tests of real performance of the (μ, LPM) - portfolio model. Harlow (1991:28-40) tested out-of-sample perfomance of the global portfolio with eleven mature capital markets and came to the conclusion that the (μ, LPM) - portfolios achieved not only higher average return but also decrease in risk measured. Sortino and Price (1994:59-64) and Nawrocki (1992:195-209) worked on the optimisation algorithm with LPM-matrix. Stevenson (2001:50-66) studied the out-of-sample performance of minimum risk and tangency portfolios and showed that only minimum LPM – portfolios consistently outperform the benchmark. Morton, Popova and Popova (2006:503-518) studied portfolio allocation in which the underlying investment instruments are hedge funds, while considering a family of utility functions involving the probability of outperforming a benchmark and expected regret relative to another benchmark. Non-normal return vectors with prescribed marginal distributions and correlation structure were modelled and simulated using the normal-to-anything method. Daníelsson et al. (2006:202-208) used regular variation to define heavy tailed distributions and showed that prominent downside risk measures produce similar and consistent ranking of heavy tailed risk. Thus, the authors concluded that regardless of the particular risk measure being used, assets are to be ranked in a similar and consistent manner for heavy tailed assets. Vercher, Bermúdez and Segura (2007:769-782) developed two fuzzy portfolio selection models, where the objective was to minimise the downside risk constrained by a given expected return. The authors assume that the rates of returns on securities are approximated as LR-fuzzy numbers of the same shape, and that the expected return and risk are evaluated by interval-valued means, so that the relationship between those mean-interval definitions for a given fuzzy portfolio by using suitable ordering relations were established. Pinar (2007:295-309) developed and tested multistage portfolio selection models maximising expected end-of-horizon wealth, while minimising one-sided deviation from a target wealth level, and report that the robust investment policies are stable in the face of market risk, while ensuring expected wealth levels quite similar to the competing expected value maximising stochastic programming model at the expense of solving larger linear programs. Bali, Demirtas and Levy (2009:883-909) examined the intertemporal relation between downside risk and expected stock returns, while using Value at Risk, Expected Shortfall, and tail risk as measures of downside risk to determine the existence and significance of a risk-return trade-off, and found a positive and significant relation between downside risk and the portfolio returns on NYSE/AMEX/Nasdaq stocks. Liang and Park (2010:199-222) compared downside risk measures that incorporate higher return moments with traditional risk measures such as standard deviation in predicting hedge fund failure. When controlling for investment strategies, performance, fund age, size, lockup, high-water mark, and leverage, they found that

[1] *Lower Partial Moment*s is one of the downside risk measures, therefor notation *Lower Partial Moments* (LPM) and downside risk measures are going to be used as equivalents.

funds with larger downside risk have a higher hazard rate. However, standard deviation loses the explanatory power once the other explanatory variables are included in the hazard model.

In conclusion it is worth to mention that downside risk measures nevertheless their long history still play important role in the literature on the field of finance and financial portfolio management, so that investigations in the current paper are going to contributed to the research.

Basic Elements of Lower Partial Moments and Their Application in Asset Allocation Process

When investment objective is defined as the aspiration return level, risk is measured in the downside part of return distribution as falling below this aspiration return τ. General continuous form of such risk measures is obtained by the evaluation of downside return deviations from the reference point τ by the general function. Bawa (1975:95-121); Fishburn (1977:116-126) and Harlow (1991:28-40), in developing the relationship between LPM and stochastic dominance, define a-degree LPM as following:

$$LPM(a;\tau) = \int_{-\infty}^{\tau} (\tau - x)^a f(x)dx \qquad [1]$$

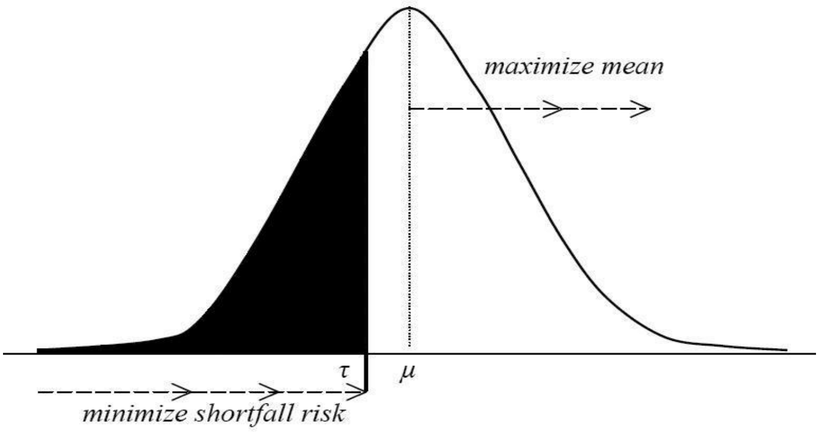

Figure 1. Optimisation with *Lower Partial Moment*
Source: worked out by the author based on Schmidt-von Rhein (2004:429)

The minimal aspiration return level divides all possible return outcomes on losses which are lower than τ, and gains which than higher than τ. Thus, its economic plausibility arises only by its correspondence with the lowest acceptable return necessary to accomplish financial goal. It is important underlining that risk measured by LPM is risk of falling below specified minimal target return denoted by x in the formula above, so that any outcomes above this reference point do not represent financial risk anymore and can be considered as chance for additional gain. Consequently the minimisation of LPM in the optimisation algorithm (discussed below) do not eliminate the chance to over-perform this reference point set as target return as it is the case in the portfolio model based on the variance (follow classical approach discussed in the previous chapter). Figure 1 shows these considerations in the graphical way.

The determination of the target return or minimum requirement level of return is difficult and a fixed defined τ as best suitable minimum level of aspiration cannot be recommended, as its size is always dependent on subjective investor-specific ideas. By setting the τ level the investor chooses critical minimum return. It should be noted that the magnitude of risk depends on the chosen τ: τ is increasing with an increasing proportion of the probability distribution seen as risky (Schmidt-von Rhein, 1996:424).

A realistic investor must be able to justify his return requirements, as the maximum return cannot always be achieved. Possible, economically justified cases for the determination of target return, which cannot be understood differently as a benchmark are the following (follow also Poddig, Brinkmann and Seiler (2005:306); Schmidt-von Rhein (1996:425-429):

- $\tau = 0$: this corresponds to determining the economic demand to safe the capital employed;

- τ = expected rate of inflation: the investor wants to achive at least a return on invested capital in the amount of the expected inflation rate for the investment period, while capital preservation is secured;
- τ = guaranteed interest rate, as by investing in the risky asset the investor loses the ability to get the risk-free interest rate;
- τ = μ: in this definition is to distinguish whether it is about return on expectations of a market index (equal to the previous case) or a risky investment (while the expected capital appreciation should be guaranteed).

Finally, it should be stressed that the increase of τ among for all LPM shows risk-increasing effect, what can be proved by differentiation with respect to τ. As it was already mentioned, the LPM in the general case can be represented as in [1]. By differentiation of the equation using the Leibniz's rule one can come to the following equation, which has a positive value:

$$\frac{\partial LPM(a;\tau)}{\partial \tau} = \int_{-\infty}^{\tau} a(\tau - x)^{a-1} f(x)dx = a \int_{-\infty}^{\tau} (\tau - x)^{a-1} f(x)dx = a * LPM(a-1;\tau) \qquad [2]$$

The minimal aspiration return τ (called also target return or benchmark) is explicitly included in the downside risk and expresses the lowest acceptable return to complete the financial target set in the beginning of the investment process as it was already mentioned. So that in the LPM framework return deviations are related to a variable investor target (follow discussion above), whereas the portfolio based on variance (in the classical approach) is related solely to the expected return. As a result, the conclusion to make is the following: the difference in the portfolio optimisation based on variance and LPM grows, the greater distance between the target and the expected return that is expressed in the shifting of the (μ, LPM) - efficient frontier further to the right in the (μ, σ) - framework.

Other important element is the order of the LPM measure. The LPM of the zero order can be defined as the probability of loss, the LPM of the first order – as the target shortfall, and finally LPM of the second order – as target semivariance. It is to be noticed that there are also target skewness and target kurtosis possible. Further details are given below.

The return deviations from the lowest target level are penalised with the α-exponent in LPM (determinating the order of the LPM) that is also an instrument expressing different degrees of risk tolerance in the asset allocation process, as it replicates the decision maker’s feelings about the relative consequences of falling below target return. Risk aversion for α >1 means that smaller losses are perceived as relatively harmful, when compared to larger losses. Risk seeking for 0<α<1 means that the main concern is to have a loss without particular regard of the amount of loss. So that the higher the difference between the α-parameter of LPM, the further the efficient frontiers are from each other, which indicates changing structure in the portfolio composition. As the variance in the return deviations are squared, the most similar efficient frontiers of the (μ, σ) portfolio model and (μ, LPM) portfolio model are obtained when the α-parameter (the order of the LPM) equals two. The more the α-parameter differs from α = 2, the more divergent the (μ, σ) - and (μ, LPM) - efficient frontiers (for further details follow Nawrocki (2003:79-96).

The LPM of the zero order is called as probability of loss (shortfall probability or target probability) and describes the occurrence possibility of an event if the minimum return requirement is exceeded. Figure 2 provides graphical description.

Such a definition of risk is an intuitively correct understanding of risk corresponding to the investors, setting the minimum return requirement. The same idea is represented by Zenger (1992:111), who claim that the concept of probability of selection is well suited because of the ease intuitive understanding of the practical implementation. At the same time the same authors point out that this concept is not applicable as the only measure of risk but in the combination. Therefore, it is important to note that not only the probability of failure, but also the extent of the failure of the investment decision is important.

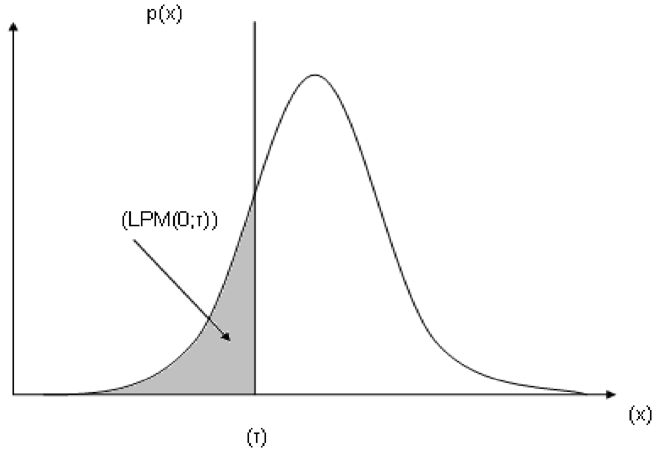

Figure 2. LPM of the Order 0 (where p(x) –probability of event; (τ) – target return; (x) – expected return)
Source: worked out by the author

It is necessary that, besides the probability of failure and the extent of this failure is measured. This can be done by using a target shortfall. This measure of risk – LPM of the order one (expected shortfall, target shortfall) measures the expected negative deviation of τ and corresponds to the LPM of the first order.

It should be noted that the combination of LPM (0, τ) and LPM (1, τ) is not a perfect solution. If it is assumed that two portfolios have the same target shortfall. Portfolio A shows a lower probability of a large loss, while Portfolio B - a high probability of a low loss (follow Figure 3). Based on the LPM (1, τ), the two portfolios are considered equally risky, while the investors sees the portfolio A as much more risky portfolio in comparison to portfolio B. It is shown that risk attitudes are not linear. High negative errors by the minimum requirements are intuitively much more weighted than the lower, the respective probability of occurrence is not taken into account. To fix this problem, the concept of semivariance is applied.

The second order LPM is determined by semivariance or downside variance and measures the expected, squared loss below the target return. Just by squaring the larger losses are weighted more heavily than the smaller ones, whereby the value of the semivariance is influenced. Thus, the second-order LPM can be seen as a risk measure, which describes the conditioned volatility below the target returns.

LPM of the third order are called semiskewness or downside skewness and fourth order as a downside curtosis or semicurtosis. In contrast to the skewness and kurtosis measures this risk measures determines only the deviation below the reference value and thus can be interpreted as follows: degree to which it comes to over-or under-proportional weighting increases below the required minimum return. Since this risk measures have less relevance in practice, they will not be considered further.

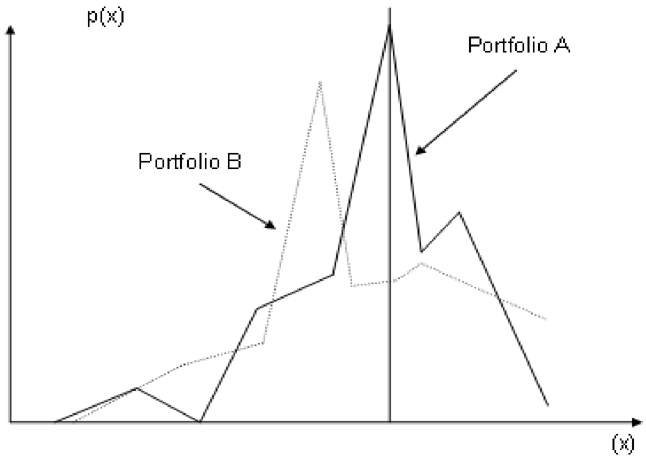

Figure 3. Two Portfolios Example (where p(x) –probability of event; (τ) – target return; (x) – expected return)
Source: developed by the author

The following table provides an overview of the LPM application in the portfolio management framework.

Table 1

Risk Measures Application in Portfolio Management

Risk measure		***Important papers***
LPM (0;τ)	Shortfall probability	Roy (1952:431-449) – the paper considers the implication of minimising the upper bound of the chance in the event of shortfall. Telser (1955:1-16) – his paper is concerned with the theory of hedging, while the investor's attitude toward risk is discussed. Kataoka (1963:181-196) – the author proposes a stochastic programming model that considers the distribution of function and probabilistic constraints. Leibowitz and Henrickson (1989:34-41) – so called confidence approach for portfolio optimisation is proposed that provides meaningful description of risk. Leibowitz, Kogelmann and Bader (1992:28-37) – the authors states that pension fund can pursue traditional asset return objectives while protecting surplus using shortfall-approach. Browne (1999:76-85) – the author argues that properties of dynamic investment strategies that minimise the probability of a shortfall relative to a given target return are misunderstood; and proposes the way that allows a decision-maker to make some definitive quantitative comparisons that are in the understanding of risk. Konno, Waki and Yuuki (2002:127-140) - the purpose of the paper is to review important characteristics of risk measures and conduct simulation using four alternative measures, lower semi-variance, lower semi-absolute deviation, first order below target risk and conditional value-at-risk, as they are useful to control downside risk when the distribution of assets is non-symmetric.
LPM (1;τ)	Shortfall expectation	Ang (1975:849-857) - presents a simple computational algorithm to approximate the E, S portfolio selection model. The essential feature of the model is the utilisation of the familiar linear programming framework by representing risks as a series of linear constraints. Yamai and Yoshiba (2005:997-1015) - in the paper the authors illustrate how the tail risk of VaR can cause serious problems in certain cases, in which expected shortfall can serve more aptly in its place. Acerbi, Nordio and Sirtori (2008:1-10) - study the properties of Expected Shortfall from the point of view of financial risk management.
LPM (2;τ)	Shortfall variance	Hogan and Warren (1972:1881-1896) – the authors suggested portfolio selection models based on expected value-semivariance criteria as it is offering certain advantages over the expected value-variance approach. Nawrocki (1999:9-25) – providing an overview about LPM development process, paying extra attention to the shortfall variance risk measure. Sing and Ong (2000:213-223) – the article demonstrates illustrates the implementation of downside risk models using spreadsheets programs. Konno, Waki and Yuuki (2002:127-140) - the purpose of the paper is to review important characteristics of risk measures and conduct simulation using four alternative measures, lower semi-variance, lower semi-absolute deviation, first order below target risk and conditional value-at-risk, as they are useful to control downside risk when the distribution of assets is non-symmetric. Estrada (2008:1-8) claims that academics and practitioners optimise portfolios using far more often the mean-variance approach than the mean-semivariance approach, and that despite the fact that semivariance is often considered a more plausible measure of risk than variance. The author proposes a heuristic approach that yields a symmetric and exogenous semicovariance matrix, which enables the determination of mean-semivariance optimal portfolios by using the well-known closed-form solutions of mean-variance problems.
LPM (3;τ)	Shortfall skewness	Harvey, Liechty, Liechty and Mueller (2004) – the authors propose a method for optimal portfolio selection using a Bayesian decision theoretic framework that addresses two major shortcomings of the Markowitz approach: the ability to handle higher moments and estimation error, while employing the skew normal distribution, which has many attractive features for modeling multivariate

		returns.
LPM (4;τ)	Shortfall kurtosis	Nawrocki (1991:465-470) – the author claims that portfolio management in the finance literature has typically used optimisation algorithms to determine security allocations within a portfolio in order to obtain the best trade-off between risk and return. These algorithms are restrictive in terms of an investor's risk aversion. Since individual investors have different levels of risk aversion, he proposes two portfolio optimisation algorithms that can be tailored to the specific level of risk aversion of the individual investor and performs ex-post evaluation tests of the algorithm performance.

Source: worked out by the author

Even though the downside measures were known for the long period of time, it is to be considered that Lower Partial Moment of the second order or semivariance are only briefly discussed in the scientific literature as appropriate risk measure in the asset allocation process, thus the main contribution of the following part is to provide to better understanding of this risk measure and show the possibilities of its practical application. So that the following optimisation problem is to be discussed Portmann (1999:87):

$$\min_{\omega}(\omega;\Delta) = \mathrm{LPM}(2;\tau) - \pi_{\mu}\omega^{T}\mu \qquad [3]$$

with $\omega^{T} = 1; \omega \geq 0; \Delta, \pi_{\mu} \in \Re$ while πμ is reduction in amount of corner portfolios. The problem could be solved under Lagrange method and Kuhn-Tucker – optimisation algorithm (for further details follow Portmann (1999, pp. 89-93)) and define the function as following:

$$\mathrm{LPM}(2;\tau) = \mathrm{LPM}(2;\tau)^{up} + \frac{\mathrm{LPM}(2;\tau)^{up} - \mathrm{LPM}(2;\tau)^{down}}{(\pi_{\mu}^{up})^2 - (\pi_{\mu}^{down})^2} \cdot \left(\left((\pi_{\mu}^{up})^2 + \frac{(\mu - (\mu_{portfolio}^{up})((\pi_{\mu}^{up}) - (\pi_{\mu}^{down}))}{\mu_{portfolio}^{up} - \mu_{portfolio}^{down}} \right)^2 - (\pi_{\mu}^{up})^2 \right) \qquad [4]$$

The other possibility is to introduce the method in general way under assumption of normal distribution of returns, while the risk measure could be expressed as following:

$$\mathrm{LPM}(2;\tau) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\tau} (\tau - x)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \qquad [5]$$

The equitation [3.4] could be standardised, while setting x equal to (σ*z+μ), if (Z ~ N(0;1)).

$$\mathrm{LPM}(2;\tau) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\frac{\tau-\mu}{\sigma}} (\tau - \mu - \sigma z)^2 e^{-\frac{z^2}{2}} dz \qquad [6]$$

In the next step the minimum frontier of the LPM of the second order is to be detrained. Thanks to the derivation of equitation [5] under the rule of Leibniz the following strictly positive result is achieved, where the relationship between LPM of the second order and σ is strictly monotone (Cremers, 2008:79-84).

$$\frac{\partial \mathrm{LPM}(2;\tau)}{\partial \sigma} = \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\tau-\mu}{\sigma}} (-z)(\tau - \mu - \sigma z) e^{-\frac{z^2}{2}} dz = \frac{2(\mu - \tau)}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\tau-\mu}{\sigma}} z e^{-\frac{z^2}{2}} dz + \frac{2\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\tau-\mu}{\sigma}} z^2 e^{-\frac{z^2}{2}} dz$$

$$\frac{\partial LPM(2;\tau)}{\partial\sigma} = \frac{2}{\sqrt{2\pi}}\left((\mu-\tau)\left[-e^{-\frac{z^2}{2}}\right]_{-\infty}^{\frac{\tau-\mu}{\sigma}} + \sigma\int_{-\infty}^{\frac{\tau-\mu}{\sigma}} z\cdot ze^{-\frac{z^2}{2}}dz\right) =$$

$$= \frac{2}{\sqrt{2\pi}}\left((\mu-\tau)\left[-e^{-\frac{z^2}{2}}\right]_{-\infty}^{\frac{\tau-\mu}{\sigma}} + \sigma\left(\left[z\left(-e^{-\frac{z^2}{2}}\right)\right]_{-\infty}^{\frac{\tau-\mu}{\sigma}}\right) - \int_{-\infty}^{\frac{\tau-\mu}{\sigma}} -e^{-\frac{z^2}{2}}dz\right) = \quad [7]$$

$$= \frac{2}{\sqrt{2\pi}}\left((\mu-\tau)\left(-e^{\frac{(\tau-\mu)^2}{2\sigma^2}}\right) + (\tau-\mu)\left(-e^{\frac{(\tau-\mu)^2}{2\sigma^2}}\right) + \sigma\int_{-\infty}^{\frac{\tau-\mu}{\sigma}} e^{-\frac{z^2}{2}}dz\right) = \frac{2\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\tau-\mu}{\sigma}} e^{-\frac{z^2}{2}}dz =$$

$$= \frac{2\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\tau} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx > 0$$

Interesting considerations on the field of portfolio optimisation were provided by Merton (1972:1851-1872): there is an analytic relationship between expected portfolio's return and standard deviation, which leads to the following equation:

$$\sigma = \sqrt{\frac{c\mu(R)^2 - 2b\mu(R) + a}{d}} \quad [8]$$

where $a = \mu^T \sum^{-1}\mu,\ b = \mu^T \sum^{-1}\iota,\ c = i^T \sum^{-1}\iota,\ d = ac\text{-}b^2$

Using Merton's equation instead of σ in [5], we come to the following functional relationship between expected return and minimum semi-variance rate in accordance to fixed target return:

$$LPM(2;\tau) = \sqrt{\frac{ac-b^2}{2\pi(c\mu^2 - 2b\mu + a)}}\int_{-\infty}^{\tau}(\tau-x)^2 e^{-\frac{(ac-b^2)(x-\mu)^2}{2(c\mu^2-2b\mu+a)}}dx =$$

$$= \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\frac{(\tau-\mu)(\sqrt{ac-b^2})}{\sqrt{c\mu^2-2b\mu+a}}} (\tau-\mu-\sqrt{\frac{c\mu^2-2b\mu+a}{ac-b^2}}z)^2 \cdot e^{-\frac{1}{2}z^2}dz \quad [9]$$

Thus the portfolio with lower semi-variance bourdon can be presented graphically as following in the figure 4. Minimum LPM(2; τ) can be determined through numeric approximation (Cremers, 2008:89):

$$\frac{\partial LPM(2;\tau)}{\partial\mu} = \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\frac{(\tau-\mu)*\sqrt{d}}{\sqrt{c*\mu^2-2b*\mu+a}}} (\mu-\tau+\frac{z^2(c*\mu-b)}{d}$$

$$+\frac{z(c*\mu^2-(3b+c\tau)*\mu+a+b*\tau)}{\sqrt{d(c*\mu^2-2b*\mu+a)}})e^{-0.5*z^2dz} \stackrel{!}{=} 0 \quad [10]$$

and weights of assets in the portfolio are determined as following:

$$\omega = \frac{\sum^{-1}\mu}{d}(c\mu-b) - \frac{\sum^{-1}\iota}{d}(b\mu-a) \quad [11]$$

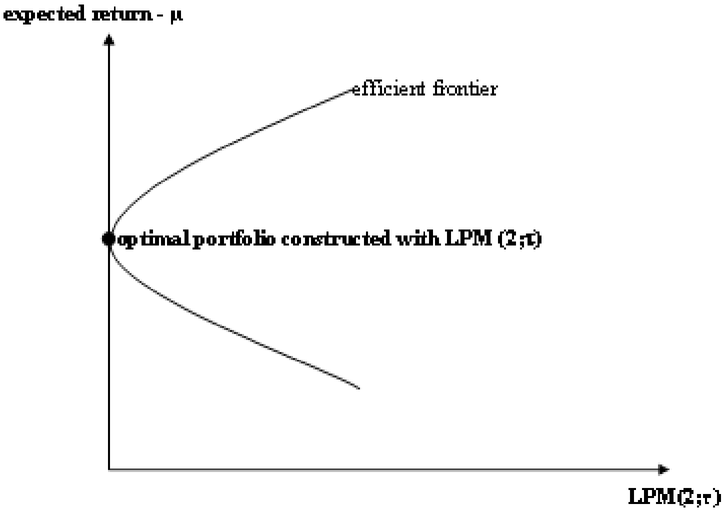

Figure 4 Portfolio Construction with Semivariance
Source: worked out by the author

As far as the possibility to invest under risk-free rate should be considered in the optimisation process, it would lead to the discussion about the construction of capital market line. Capital market line is a line used in the capital asset pricing model to illustrate the rates of return for efficient portfolios depending on the risk-free rate of return and the level of risk for a particular portfolio. In order to determine the capital market line and detect the optimal portfolio expected return of the market portfolio and expected return of the portfolio under optimisations are to be determined.

In order to determine efficient portfolio with LPM of the second order the following considerations are to be considered: the starting point in this discussion is the concept of corner portfolio - is optimal portfolio for a given risk tolerance at which a variable changes status. It is called a corner portfolio because in a graph that plots asset holdings – asset weights against risk tolerance – lambda, two or more variables turn a corner.

In the framework of portfolio optimisation this question was discussed by Serf (1995:177-184) and based on his considerations the return of the corner portfolio can be seen as linear combination:

$$\mu = \alpha\mu^{up} + (1+\alpha)\mu^{down} \qquad [12]$$

where α is weights – factor that shows the following characteristics:

$$\alpha \in [0;1],\ \alpha = 0\Big|_{\mu=\mu^{down}}$$

$$\alpha = 1\Big|_{\mu=\mu^{up}} .$$

According to the previous equation the variance of the portfolio and weights of assets in the portfolio could be established:

$$\sigma^2 = \omega^T \sum \omega = (\alpha\varpi_{up} + (1-\alpha)\omega_{down})^T \sum (\alpha\omega_{up} + (1-\alpha)\omega_{down}) \qquad [13]$$

$$\omega^T = \left(\frac{\mu - \mu^{down}}{\mu^{up} - \mu^{down}}(\omega_{up} - \omega_{down}) + \omega_{down}\right) \qquad [14]$$

In the concept of LPM of the second order, while looking for an efficient portfolio by utilisation of equations [12] to [14], the standard deviation term – σ in the equation [5] is to be changed against [14] in order to find the relationship between expected return of the portfolio and shortfall variance, and thus to find efficient portfolio (Kaduff, 1996:167):

$$LPM(2;\tau)=\sqrt{\frac{1}{2\pi(\omega_{portfolio}^{T}\sum\omega_{portfolio})}\int_{-\infty}^{\tau}(\tau-x)^{2}e^{-\frac{(x-\mu)^{2}}{2\omega_{portfolio}^{T}\sum\omega_{portfolio}}}dx} \qquad [15]$$

with following weights of assets in the portfolio:

$$\omega_{portfolio}^{T}=\left(\frac{\mu-\mu^{down}}{\mu^{up}-\mu^{down}}\left[\omega_{up}-\omega_{down}\right]+\omega_{down}\right) \qquad [16]$$

The purpose of the theoretical overview was to introduce the LPM of the second order as an appropriate risk measure in the portfolio optimisation process.

Usage of Lower Partial Moments in Management of Investment Portfolios: Theoretical Description

The importance of asset allocation and portfolio management for the Latvian insurance companies to cover losses from their main business operations were mentioned in the in the introductory part of the current paper. The asset allocation decision is not an isolated choice, but rather a component of a structured four-step portfolio management process that never stops. Due to the importance of the topic, the process of portfolio management will be discussed in detail below. The author will also include a portfolio construction algorithm with *Lower Partial Moments* (LPM) in this process.

Taking into consideration the fact that in the process of financial portfolio construction, while making decisions about an investment, investors are more concerned with the downside movements of their portfolios (when their target return has failed), than with the upside potential, the main concern of that particular algorithm is in regard to the possibilities of asset allocation based on downside risk. The existence of– risk measure, presenting the squared failure of the investor's target return can be justified by its dominance *Lower Partial Moment* being equivalent to the stochastic dominance of the third order. Thus, the dominance concerning *Lower Partial Moments* of the second order fulfils the criterion for the Bernoulli principle and therefore can be used for decision making under risk (Estrada, 2008:9-11).

The first step in the portfolio management process for the insurance company's management (either using internal resources – an in-house team or with the assistance of an external investment advisor) is to construct a policy statement. The policy statement should be understood as a road map, where the investor specifies the types of risk he is willing to take (by determination of a risk aversion parameter – α), investment goals (capital preservation, capital appreciation, current income by determination of target return parameter - τ) and constraints (like liquidity, time horizon, tax concerns, legal and regulatory requirements, etc.). Since investor needs change over time, the policy statement must be periodically reviewed and updated. The process of investment seeks to look into the future and determine strategies that offer the best possibility of meeting the policy statement guidelines determined in the previous step.

In the second step of the process – determination of financial strategy – the management should study current financial and economic conditions and forecast future trends, which require constant monitoring and updating to be able to reflect changes in financial market expectations.

The third step of the portfolio management process is to construct the actual portfolio. With the policy statement and financial market forecast as input, implementation of the investment strategy is prepared by determining asset allocation across countries, asset classes and different securities. Portfolio construction is achieved by minimising risk and maximising expected return. When comparing the classical (μ,σ) model and the (μ, LPM) model, it should be said that the models differ substantially in terms of risk measurement and return dependence. The (μ,σ) model only takes into account the mean and variance, whereas the (μ, LPM) model also considers non-normality of return distribution. As the same input data were used for both portfolio models, which differ only in risk measure, an enhanced realised performance can be explained only by the application of more appropriate and exact risk measures. In the construction process of a tangential portfolio on (μ, LPM) – an efficient frontier should be chosen. The tangential portfolio holds the maximum efficiency portfolio: the portfolio with the highest return premium on one unit of risk.

The last step is the continual monitoring of the needs and capital market conditions. One of the essential components of the monitoring process is the evaluation of the portfolio's performance and its comparison to the goals set in the policy statement.

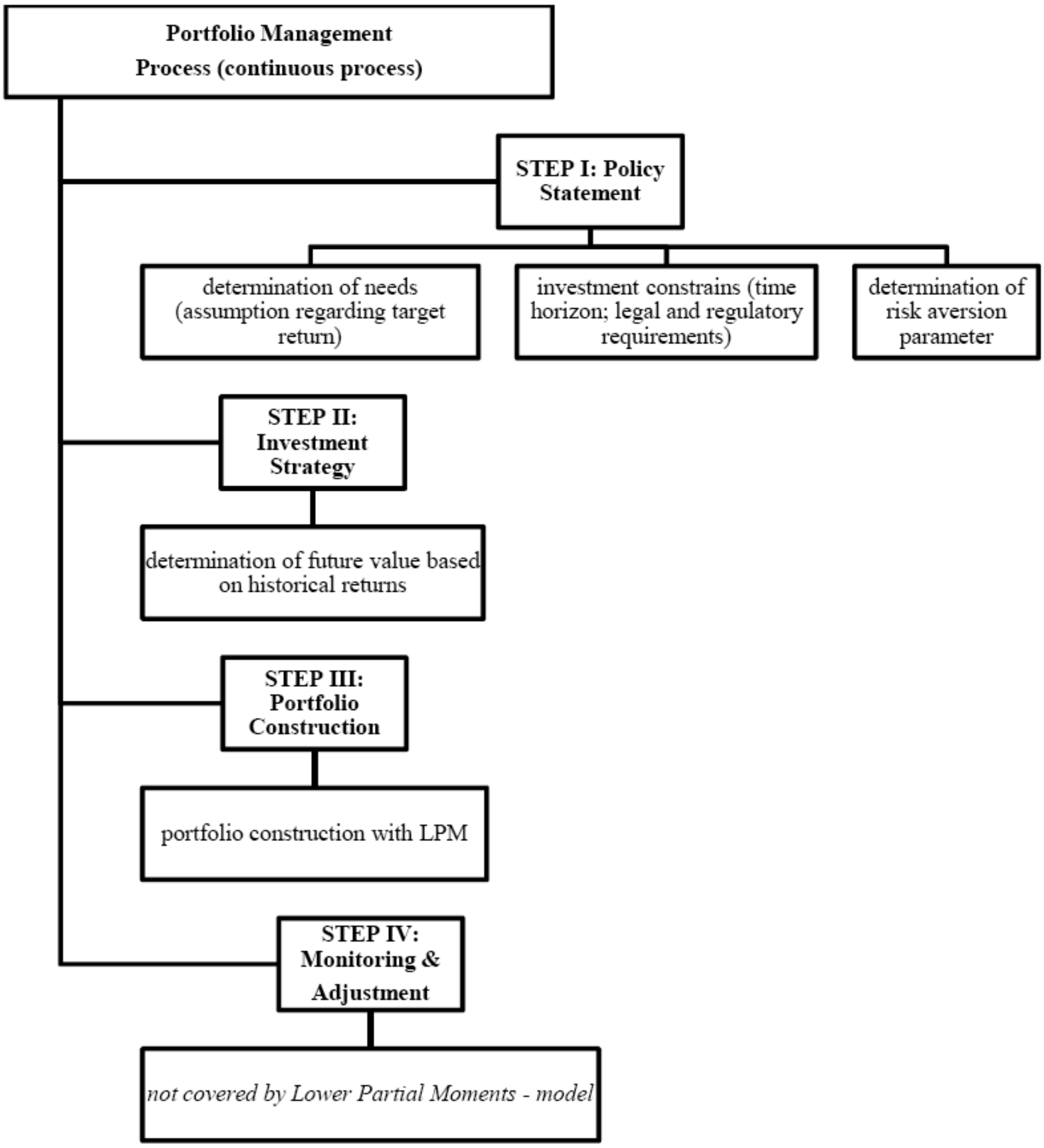

Figure 5. Portfolio Management Process
Source: worked out by the author

Due to the relevance of this topic for practical implementation further research on the impact of the two portfolio models representing the two approaches in the portfolio theory – classical model and model based on the LPM - should be examined. Further considerations can be found in Kuzmina (2011:361-372).

Conclusions

Main goal of the current research was to present financial portfolio management model as an internal model for insurance companies holding small number of stocks in their investment portfolios, which not only satisfies regulatory requirements and internal risk management standards, but also allows dealing with otherwise complex multivariate modelling using generally available computation applications, due to the fact that so called "all in one solutions" like for example BARRA, NORTHFIELD, WILSHIRE and others require considerable financial investments and present a kind of black box (as several estimation parameters and computation techniques are not completely disclosed).

The reader was instructed on the essential aspects of the (μ,LPM)- portfolio model which, on the one side, enables its critical review, and on the other side, provides a platform for its later application in the practice of portfolio management. The research was concerned with the portfolio selection based on the downside risk and mean, which utilises risk measure corresponding with the risk understanding of the prevailing number of investors. As a consequence, by the portfolio optimisation based on the downside risk the chance to over-perform the reference point is not minimised as by the portfolio optimisation based on the variance.

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